

Entire f -maximal graphs in the lorentzian product $\mathbb{G}^n \times \mathbb{R}_1$

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Abstract

In the lorentzian product $\mathbb{G}^n \times \mathbb{R}_1$, we give a comparison between the f -volume of an entire f -maximal graph and the f -volume of the hyperbolic H_r^+ under the assumption that the gradient of the function defining the graph is bounded away from 1. As a consequence, we obtain a Calabi-Bernstein type theorem for f -maximal graphs in $\mathbb{G}^n \times \mathbb{R}_1$. Without the condition on the gradient of the function, an example of non-planar entire f -maximal graph in the lorentzian product $\mathbb{G}^2 \times \mathbb{R}_1$ is given. This example shows that the assumption on the gradient of the function defining the graph in the volume comparison as well as in the Bernstein type theorem is essential.

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1 Introduction

Let $\Gamma \subset \mathbb{R}^3$ be a simple closed smooth curve and Σ be a smooth surface with $\partial\Sigma = \Gamma$. If for all deformations Σ_t , where $\Sigma_0 = \Sigma$ and $\partial\Sigma_0 = \partial\Sigma = \Gamma$, Σ has smallest area, i.e.

$$\text{Area}(\Sigma_t) \geq \text{Area}(\Sigma);$$

then it follows that

$$\frac{d}{dt} \text{Area}(\Sigma_t)|_{t=0} = 0, \quad (1)$$

and

$$\frac{d^2}{dt^2} \text{Area}(\Sigma_t)|_{t=0} \geq 0. \quad (2)$$

A surface Σ is said to be minimal if (1) holds and moreover is said to be stable if (2) holds. It is well known that a surface is minimal if and only if its mean curvature is zero everywhere. If Σ is the graph of a function $u(x, y)$ over a domain D , then u must satisfy the Lagrange equation

$$(1 + u_x^2)u_{yy} - 2u_xu_yu_{xy} + (1 + u_y^2)u_{xx} = 0. \quad (3)$$

Dirichlet problem for Equation (3) with a boundary condition has a uniquely solution. So there is a rich family of minimal graphs over a disk of radius R . When R tends to infinity, one would expect with a suitable choice of boundary conditions, to construct many minimal graphs over the whole plane \mathbb{R}^2 . But Bernstein (1915-1917) proved a surprise theorem, named after him, that a minimal graph over \mathbb{R}^2 must be a plane. For a long time, it was conjectured that Bernstein's theorem holds for minimal graphs over \mathbb{R}^n for $n > 2$. For $n = 3$, the theorem is proved by De Giorgi [12], for $n = 4$ by F. Almgren [6] and for $n = 5, 6, 7$ by J. Simons [19]. For $n \geq 8$, it is proved that there are entire minimal graphs other than the hyperplanes by E. Bombieri, E. De Giorgi and E. Giusti [7].

The lorentzian version of the theorem for maximal graphs in Lorent-Minkowski spaces is called Calabi-Bernstein's Theorem. The theorem was first proved by Calabi ([8]) for the case $n = 2$ and later by Cheng and Yau ([10]) for the case of general dimensions. The theorem has been generalized in some lorentzian product spaces ([2], [3], [4], [5], [17]) and in other ambient spaces such as warped product, Robertson-Walker spacetimes, manifolds with density... ([9], [13], [16], [18], [20]).

L. Wang [20] prove a Bernstein type theorem for self-shrinkers in \mathbb{R}^n . Self shrinkers are simplest solutions to the mean curvature flow that satisfy the following equation

$$H = \langle x, N \rangle,$$

where N is the unit normal vector field and $H = \operatorname{div} N$ is the mean curvature, are just minimal hypersurfaces in \mathbb{R}^n under the conformally changed metric $g_{ij} = e^{-\frac{|x|^2}{2n}} \delta_{ij}$ (see [11]) or special examples of f -minimal hypersurfaces in \mathbb{R}^n with density $e^{-\frac{|x|^2}{4}}$, a modified version of Gauss space.

The last two authors ([13]) proved a Bernstein type theorem for entire f -minimal graph Σ over \mathbb{G}^n , where \mathbb{G}^n is Gauss space, that is \mathbb{R}^n endowed with Gaussian density $e^{-f} = e^{c-\frac{|x|^2}{2}}$, where $c = \log[(2\pi)^{-\frac{n}{2}}]$. By the Fundamental Theorem of Calibrated Geometry $\Sigma_R = \Sigma \cap B(p, R)$ is f -volume minimizing and therefore, an estimation of the f -volume of Σ_R is established. Because the density decays at infinity faster than any power of R , taking $R \rightarrow \infty$, they conclude that $\operatorname{Vol}_f(\Sigma) = \operatorname{Vol}_f(\mathbb{G}^n) = 1$. This important fact leads to a simple proof of the theorem without using second order differential equations.

So it is naturally to use the same idea to prove the lorentzian version of the result, i.e. a Calabi-Bernstein type theorem for entire f -maximal graphs in the lorentzian product $\mathbb{G}^n \times \mathbb{R}_1$. The situation is quite different. In this space, the spacelike pseudo hyperspheres H_r^+ are unbounded, therefore the intersection $\Sigma_r = \Sigma \cap H_r^+$ may be unbounded and the same calibration argument can not be used. To avoid this, the gradient of the function determining the graph is assumed to be bounded away from 1. With this additional assumption, the f -volume of Σ is proved less than f -volume of H_r^+ for any r . Taking $r \rightarrow \infty$ we obtain $\operatorname{Vol}_f(\Sigma) = \operatorname{Vol}_f(\mathbb{G}^n) = 1$ and therefore the theorem is proved.

Finally, to ensure that the condition on the gradient of the function defining the graph in the volume comparison as well as in the Calabi-Bernstein type theorem is essential, we construct an example of non-planar entire f -maximal graph in the lorentzian product $\mathbb{G}^2 \times \mathbb{R}_1$. It should be mentioned that, some non-planar entire maximal graphs in the lorentzian product $\mathbb{H}^2 \times \mathbb{R}_1$ were found in [1].

2 Preliminaries

2.1 The Lorentz-Minkowski space \mathbb{R}_1^{n+1}

The Lorentz-Minkowski space \mathbb{R}_1^{n+1} is the $(n+1)$ -dimensional vector space $\mathbb{R}^{n+1} = \{(x_1, x_2, \dots, x_{n+1}) : x_i \in \mathbb{R}, i = 1, 2, \dots, n+1\}$ with the pseudo scalar product given by

$$\langle \mathbf{x}, \mathbf{y} \rangle = \sum_{i=1}^n x_i y_i - x_{n+1} y_{n+1},$$

where $\mathbf{x} = (x_1, x_2, \dots, x_{n+1})$, $\mathbf{y} = (y_1, y_2, \dots, y_{n+1}) \in \mathbb{R}^{n+1}$. Since $\langle \cdot, \cdot \rangle$ is non-positive definite, $\langle \mathbf{x}, \mathbf{x} \rangle$ may be zero or negative. A nonzero vector $\mathbf{x} \in \mathbb{R}_1^{n+1}$ is called spacelike, lightlike or timelike if $\langle \mathbf{x}, \mathbf{x} \rangle > 0$, $\langle \mathbf{x}, \mathbf{x} \rangle = 0$ or $\langle \mathbf{x}, \mathbf{x} \rangle < 0$, respectively. If $\langle \mathbf{x}, \mathbf{y} \rangle = 0$; \mathbf{x}, \mathbf{y} are said to be pseudo orthogonal. The norm of a vector \mathbf{x} is defined by $\|\mathbf{x}\| = \sqrt{|\langle \mathbf{x}, \mathbf{x} \rangle|}$.

For $\mathbf{x}_i = (x_1^i, x_2^i, \dots, x_n^i, x_{n+1}^i)$, $i = 1, 2, \dots, n$; we define the pseudo vector product $\mathbf{x}_1 \wedge \mathbf{x}_2 \wedge \dots \wedge \mathbf{x}_n$ by

$$\mathbf{x}_1 \wedge \mathbf{x}_2 \wedge \dots \wedge \mathbf{x}_n = \begin{vmatrix} \mathbf{e}_1 & \mathbf{e}_2 & \cdots & \mathbf{e}_n & -\mathbf{e}_{n+1} \\ x_1^1 & x_2^1 & \cdots & x_n^1 & x_{n+1}^1 \\ x_1^2 & x_2^2 & \cdots & x_n^2 & x_{n+1}^2 \\ \vdots & \vdots & \cdots & \vdots & \vdots \\ x_1^n & x_2^n & \cdots & x_n^n & x_{n+1}^n \end{vmatrix},$$

where $\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_n, \mathbf{e}_{n+1}$ is the canonical basis of \mathbb{R}_1^{n+1} . It is clear that, for every $\mathbf{x} \in \mathbb{R}_1^{n+1}$,

$$\langle \mathbf{x}, \mathbf{x}_1 \wedge \mathbf{x}_2 \wedge \dots \wedge \mathbf{x}_n \rangle = \det(\mathbf{x}, \mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_n),$$

and therefore $\mathbf{x}_1 \wedge \mathbf{x}_2 \wedge \dots \wedge \mathbf{x}_n$ is pseudo orthogonal to any \mathbf{x}_i , $i = 1, 2, \dots, n$.

For more information about Lorentz-Minkowski spaces, we refer the reader to [14] and [15].

A k -dimensional surface $\Sigma \in \mathbb{R}_1^{n+1} = \mathbb{R}^n \times \mathbb{R}_1$, $k \leq n$ is called spacelike if the restriction $\langle \cdot, \cdot \rangle_\Sigma$ is positive definite or equivalently, every tangent vector of Σ is spacelike. A spacelike hypersurface is called maximal if its mean curvature $H = \operatorname{div} N$ is zero everywhere, here N stands for a unit timelike normal vector field of Σ . If a spacelike hypersurface Σ is the graph of a function $u : \mathbb{R}^n \rightarrow \mathbb{R}_1$, then Σ is maximal if and only if u satisfies the maximal equation

$$\operatorname{div} N = \operatorname{div} \left(\frac{\nabla u}{\sqrt{1 - |\nabla u|^2}} \right) = 0.$$

2.2 f -maximal hypersurfaces

Now suppose that on \mathbb{R}_1^{n+1} there is a positive function e^{-f} , called the density, used to weight volume of k -dimensional spacelike surfaces. Such space \mathbb{R}_1^{n+1} is called a Lorentz-Minkowski space with density. The weighted volume of a spacelike hypersurface Σ , denoted by $\operatorname{Vol}_f(\Sigma)$, in \mathbb{R}_1^{n+1} with density e^{-f} is defined by

$$\operatorname{Vol}_f(\Sigma) = \int_{\Sigma} e^{-f} dV,$$

where dV is the volume element of the hypersurface. The weighted mean curvature or f -mean curvature of Σ , denoted by H_f , is defined by

$$H_f = H - \langle \nabla f, N \rangle.$$

The hypersurface Σ is called f -maximal if $H_f = 0$ everywhere, i.e. $H = \langle \nabla f, N \rangle$.

If Σ is the graph of a function $u : \mathbb{R}^n \rightarrow \mathbb{R}_1$, then Σ is f -maximal if and only if u satisfies the f -maximal equation

$$\operatorname{div} N - \langle \nabla f, N \rangle = \operatorname{div} \left(\frac{\nabla u}{\sqrt{1 - |\nabla u|^2}} \right) - \langle \nabla f, N \rangle = 0.$$

3 Entire f -maximal graph in $\mathbb{G}^n \times \mathbb{R}_1$

3.1 Bernstein type theorem for entire f -maximal graphs in $\mathbb{G}^n \times \mathbb{R}_1$

The space $\mathbb{G}^n \times \mathbb{R}_1$ is just $\mathbb{R}_1^{n+1} = \mathbb{R}^n \times \mathbb{R}_1$ endowed with the Euclidean-Gaussian density

$$e^{-f(\mathbf{x}, t)} = e^{c - \frac{|\mathbf{x}|^2}{2}},$$

where $c = \log[(2\pi)^{-\frac{n}{2}}]$, $(\mathbf{x}, t) \in \mathbb{G}^n \times \mathbb{R}_1$, $\mathbf{x} \in \mathbb{G}^n$, $t \in \mathbb{R}_1$. Denote by:

- $H_r = \{(\mathbf{x}, t) \in \mathbb{G}^n \times \mathbb{R}_1 : \langle \mathbf{x}, \mathbf{x} \rangle - \langle t, t \rangle = -r^2\}$: the hyperbolic with center O and radius r ;
- $H_r^+ = \{(\mathbf{x}, t) \in \mathbb{G}^n \times \mathbb{R}_1 : \langle \mathbf{x}, \mathbf{x} \rangle - \langle t, t \rangle = -r^2, t > 0\}$;
- $LC = \{(\mathbf{x}, t) \in \mathbb{G}^n \times \mathbb{R}_1 : \langle \mathbf{x}, \mathbf{x} \rangle - \langle t, t \rangle = 0\}$: the lightcone;
- B_R^n : the n -ball in \mathbb{G}^n with center O and radius R ;
- $\mathcal{C} = B_R^n \times \mathbb{R}$: the $(n+1)$ -cylinder over B_R^n ;
- $S_R^{n-1} = \partial B_R^n$: the $(n-1)$ -sphere in \mathbb{G}^n with center O and radius R .

In this section, let Σ be the graph of a function $u(\mathbf{x}) = t$ over \mathbb{G}^n that is f -maximal. Let N be the future-pointing unit timelike normal vector field of Σ and consider the smooth extension of N by the translation along t -axis, also denoted by N .

Let w be the n -differential form defined by

$$w(X_1, X_2, \dots, X_n) = -\langle X_1 \wedge X_2 \wedge \dots \wedge X_n, N \rangle = -\det(X_1, X_2, \dots, X_n, N), \quad (4)$$

where X_i , $i = 1, 2, \dots, n$ are smooth vector fields, $X_1 \wedge X_2 \wedge \dots \wedge X_n$ is the pseudo vector product and $\langle \cdot, \cdot \rangle$ is the pseudo scalar product. It is not hard to prove the following.

Lemma 1. *The form w is a lorentzian calibration that calibrates Σ , i.e.*

1. $d(e^{-f}w) = 0$;
2. $|w(X_1, X_2, \dots, X_n)| \geq 1$, for every orthonormal spacelike vector fields X_i , $i = 1, 2, \dots, n$ and the equality holds if and only if X_1, X_2, \dots, X_n are tangent to Σ . Moreover,

$$w(X_1, X_2, \dots, X_n) = 1$$

if $X_1 \wedge X_2 \wedge \dots \wedge X_n$ is in the opposite direction to N .

Proof. 1. We have

$$d(e^{-f}w) = e^{-f} \operatorname{div} N - e^{-f} \langle \nabla f, N \rangle = e^{-f} (\operatorname{div} N - \langle \nabla f, N \rangle).$$

Because Σ and all its translations along the t -axis are f -maximal, $d(e^{-f}w) = 0$.

2. Since X_1, X_2, \dots, X_n are spacelike, $X_1 \wedge X_2 \wedge \dots \wedge X_n$ is timelike. Therefore

$$|w(X_1, X_2, \dots, X_n)| \geq 1.$$

The equality holds if and only if $X_1 \wedge X_2 \wedge \dots \wedge X_n$ and N are parallel, i.e. X_1, X_2, \dots, X_n are all tangent to Σ . Moreover, if $X_1 \wedge X_2 \wedge \dots \wedge X_n$ is in the opposite direction to N , $w(X_1, X_2, \dots, X_n) = 1$. □

By Stokes' Theorem, as in the case of f -minimal surfaces (see [13]), it is proved that Σ is f -area-maximizing, i.e. any compact portion of Σ has largest area among all spacelike surfaces in its homology class.

The next lemma gives a comparison, when the Jacobian ∇u is bounded away from 1, between the f -volume of Σ and the f -volume of $H^+(r)$.

Lemma 2. *If the Jacobian ∇u is bounded away from 1, then for every $r > 0$,*

$$\text{Vol}_f(\Sigma) \geq \text{Vol}_f(H_r^+).$$

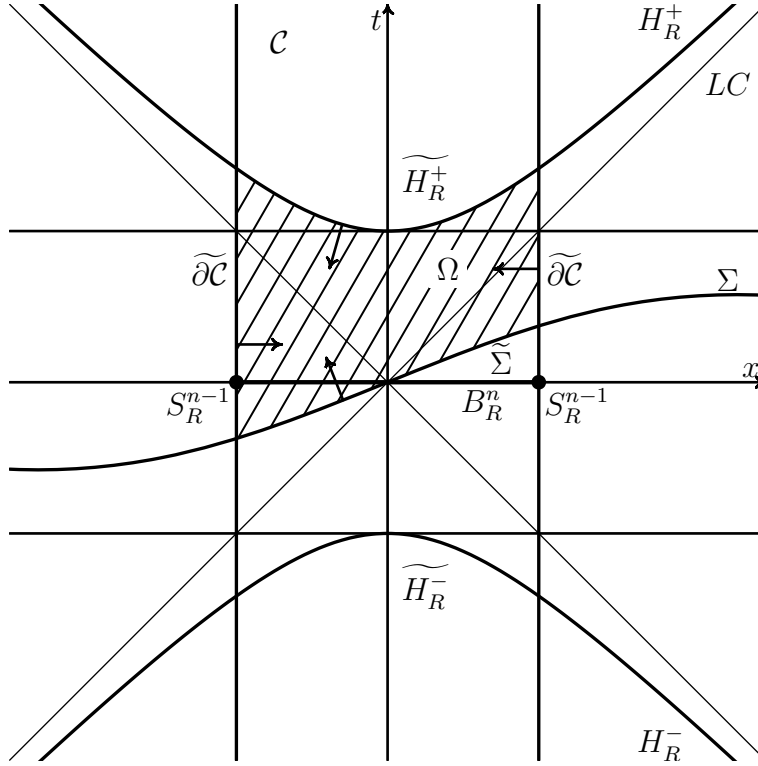


Figure 1: Proof of Lemma 2

Proof. Since the density does not depend on the last coordinate, we can assume that the origin $O \in \Sigma$. Denote $\Sigma \cap \mathcal{C} := \tilde{\Sigma}$ and $H_r^+ \cap \mathcal{C} := \tilde{H}_r^+$. Since Σ is spacelike, $\tilde{\Sigma}$ is bounded and is in the slab between parallel hyperplanes $t = \pm R$. Denote by

- Ω be the region bounded by $\mathcal{C}, \tilde{\Sigma}, \tilde{H}_r^+$ with inward-pointing orientation;

- $\widetilde{\partial\mathcal{C}}$ be the part of $\partial\mathcal{C}$ between $\widetilde{\Sigma}$ and H_r^+ ;
- w be the lorentzian calibration constructed as above.

Note that, the directions of both Σ and H_r^+ are future-pointing, by Stokes' Theorem, we get

$$0 = \int_{\Omega} e^{-f} dw = \int_{\partial\Omega} e^{-f} w = \int_{\widetilde{\Sigma}} e^{-f} w - \int_{\widetilde{H_r^+}} e^{-f} w + \int_{\widetilde{\partial\mathcal{C}}} e^{-f} w,$$

or

$$\int_{\widetilde{H_r^+}} e^{-f} w = \int_{\widetilde{\Sigma}} e^{-f} w + \int_{\widetilde{\partial\mathcal{C}}} e^{-f} w.$$

By Lemma 1, $\int_{\widetilde{H_r^+}} e^{-f} w \geq \text{Vol}_f(\widetilde{H_r^+})$ and $\int_{\widetilde{\Sigma}} e^{-f} w = \text{Vol}_f(\widetilde{\Sigma})$. Thus, we obtain the following estimate

$$\text{Vol}_f(\widetilde{H_r^+}) \leq \text{Vol}_f(\widetilde{\Sigma}) + \int_{\widetilde{\partial\mathcal{C}}} e^{-f} w.$$

Since $\lim_{R \rightarrow \infty} \text{Vol}_f(\widetilde{H_r^+}) = \text{Vol}_f(H_r^+)$ and $\lim_{R \rightarrow \infty} \text{Vol}_f(\widetilde{\Sigma}_R) = \text{Vol}_f(\Sigma)$, so the remain thing we have to prove is $\lim_{R \rightarrow \infty} \int_{\widetilde{\partial\mathcal{C}}} e^{-f} w = 0$.

Let $\overline{\partial\mathcal{C}}$ is the part of $\partial\mathcal{C}$ between two parallel hyperplane $t = \pm\sqrt{R+r}$, $\overline{\partial\mathcal{C}} = S_R^{n-1} \times [-\sqrt{R+r}, \sqrt{R+r}]$. Because $\widetilde{\partial\mathcal{C}} \subset \overline{\partial\mathcal{C}}$ and the density on $\overline{\partial\mathcal{C}}$ is the constant $e^{-f(R)}$, we get

$$\int_{\widetilde{\partial\mathcal{C}}} e^{-f} w \leq e^{-f(R)} \int_{\overline{\partial\mathcal{C}}} w.$$

Note that a normal vector of \mathcal{C} is of the form $(\mathbf{x}, 0)$, therefore

$$\int_{\overline{\partial\mathcal{C}}} w = \int_{\overline{\partial\mathcal{C}}} \frac{\langle \nabla u, \mathbf{x} \rangle}{\sqrt{1 - |\nabla u|^2}}.$$

By the assumption, the Jacobian ∇u is upper bounded away from 1, we have

$$\int_{\overline{\partial\mathcal{C}}} \frac{\langle \nabla u, \mathbf{x} \rangle}{\sqrt{1 - |\nabla u|^2}} \leq K \text{Vol}(\overline{\partial\mathcal{C}}) = 2K\sqrt{R+r} \text{Vol}(S_R^{n-1}),$$

where $K > 0$ is a constant. Finally, we get the following inequality

$$\int_{\widetilde{\partial\mathcal{C}}} e^{-f} w \leq 2e^{-f(R)} K\sqrt{R+r} \text{Vol}(S_R^{n-1}). \quad (5)$$

It is easy to see that, the right handside of (5) goes to zero when R approaches infinity and therefore the proof of the lemma is complete. \square

Theorem 3. In $\mathbb{G}^n \times \mathbb{R}_1$, let Σ be the graph of a function $u(\mathbf{x}) = t$ over \mathbb{G}^n and ∇u is bounded away from 1. Then Σ is f -maximal if and only if u is constant, i.e. Σ is a hyperplane.

Proof. Note that H_r^+ is the graph of the function $g(\mathbf{x}) = \sqrt{\mathbf{x}^2 + r^2}$, so

$$\text{Vol}_f(H_r^+) = \int_{\mathbb{R}^n} e^{-f} \sqrt{\frac{r^2}{\mathbf{x}^2 + r^2}};$$

and

$$\lim_{r \rightarrow \infty} \text{Vol}_f(H_r^+) = 1.$$

Therefore,

$$\begin{aligned} 1 &= \text{Vol}_f \mathbb{G}^n = \int_{\mathbb{R}^n} e^{-f} \\ &\geq \text{Vol}_f(\Sigma) = \int_{\mathbb{R}^n} e^{-f} \sqrt{1 - |\nabla u|^2} \\ &\geq \lim_{r \rightarrow \infty} \text{Vol}_f(H_r^+) = 1. \end{aligned}$$

The equality holds if and only if $|\nabla u|^2 = 0$, i.e. u is constant. \square

3.2 An example of non-planar entire f -maximal graph in $\mathbb{G}^2 \times \mathbb{R}_1$

Let Σ_0 be the graph of the function $u : \mathbb{R}^2 \rightarrow \mathbb{R}$, $(x, y) \mapsto \int_0^x \sqrt{\frac{e^{\tau^2}}{1 + e^{\tau^2}}} d\tau$, i.e. Σ_0 has a parametrization

$$X(x, y) = \left(x, y, \int_0^x \sqrt{\frac{e^{\tau^2}}{1 + e^{\tau^2}}} d\tau \right). \quad (6)$$

Since

$$1 - |\nabla u|^2 = \frac{1}{1 + e^{x^2}} > 0,$$

Σ_0 is spacelike. Moreover, a simple direct computation yields: the future-pointing unit timelike normal vector field is

$$N = \left(\frac{u_x}{\sqrt{1 - u_x^2}}, 0, \frac{1}{\sqrt{1 - u_x^2}} \right);$$

the coefficients of the first fundamental form are

$$E = 1 - u_x^2, \quad F = 0, \quad G = 1;$$

the coefficients of the second fundamental form are

$$e = \frac{u_{xx}}{\sqrt{1 - u_x^2}}, \quad f = g = 0;$$

the mean curvature is

$$H = \frac{eG - 2fF + gE}{(EG - F^2)} = \frac{u_{xx}}{(1 - u_x^2)^{3/2}} = xe^{x^2/2},$$

and

$$\langle \nabla f, N \rangle = \frac{xu_x}{\sqrt{1 - u_x^2}} = xe^{x^2/2}.$$

Thus,

$$H_f = H - \langle \nabla f, N \rangle = 0,$$

i.e. Σ_0 is maximal.

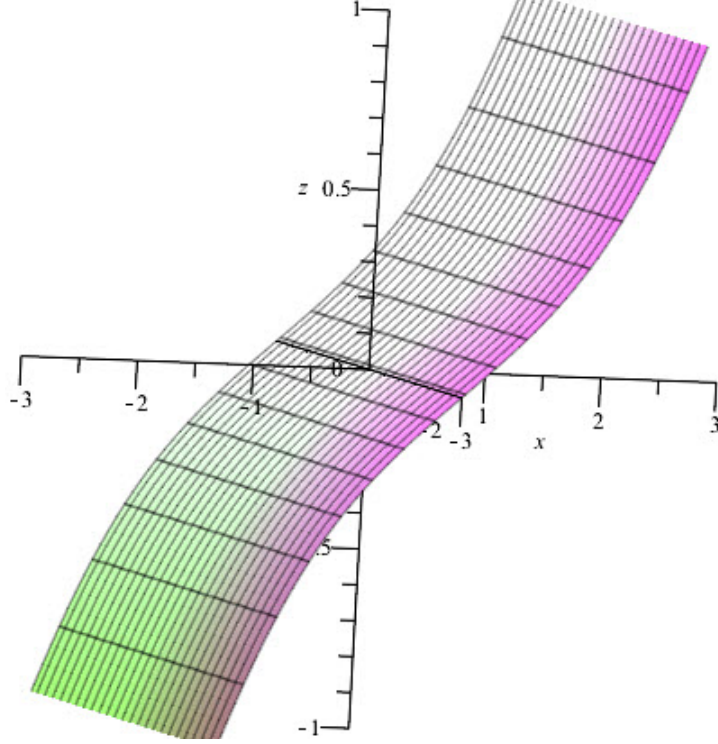


Figure 2: A non-planar entire f -maximal graph

Remark 4. The example of non-planar entire f -maximal surface Σ_0 shows that the assumption on ∇u is essential.

1. $\lim_{x \rightarrow \infty} |\nabla u|^2 = \lim_{x \rightarrow \infty} \frac{e^{x^2}}{1+e^{x^2}} = 1.$
2. Since $\lim_{r \rightarrow \infty} \text{Vol}_f(H_r^+) = 1$ and $\text{Vol}_f(\Sigma_0) < 1$, when r is large enough, $\text{Vol}_f(\Sigma_0) < \text{Vol}_f(H_r^+).$

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